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Parabolic subgroups of finite index in Coxeter groups

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Abstract

In this paper, we investigate irreducible Coxeter systems and we determine the minimal parabolic subgroup of finite index in a Coxeter group.

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1. Introduction and preliminaries

The purpose of this paper is to study parabolic subgroups of a Coxeter group. A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \rangle,$$

where S is a finite set and $m: S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:
(1) $m(s,t) = m(t,s)$ for all $s,t \in S$,
(2) $m(s,s) = 1$ for all $s \in S$, and
(3) $m(s,t) \geqslant 2$ for all $s \neq t \in S$.
The pair (W,S) is called a *Coxeter system*. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. It is known that the pair (W_T, T) is also a Coxeter system [1]. If T is the empty set, then W_T is the trivial group. A Coxeter system (W,S) is said to be *irreducible* if, for any nonempty and proper subset T of S, W does not decompose as the direct product of W_T and $W_{S \setminus T}$.

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After some preliminaries in Section 2, we show the following theorem in Section 3.

Theorem. *Let (W,S) be an irreducible Coxeter system. If W is infinite, then W has no parabolic subgroup of finite index other than W itself.*

By the above theorem we can determine the minimal parabolic subgroup of finite index in a Coxeter system. Let (W,S) be a Coxeter system. Then there exists a unique decomposition $\{S_1,\dots,S_r\}$ of S such that W is the direct product of the parabolic subgroups $W_{S_1},\dots,W_{S_r}$ and each Coxeter system (W_{S_i},S_i) is irreducible (cf. [1, 7, p. 30]). Let $\tilde{S} := \bigcup\{S_i \mid W_{S_i} \text{ is infinite}\}$. Then we obtain the following corollary.

Corollary. *The parabolic subgroup $W_{\tilde{S}}$ is the minimal parabolic subgroup of finite index in W. Hence, if W_{S_0} is a parabolic subgroup of finite index in W, then*
(1) $\tilde{S} \subset S_0$,
(2) $W_{S_0} = W_{\tilde{S}} \times W_{S_0\setminus\tilde{S}}$ *and*
(3) $[W : W_{S_0}] = |W_{S\setminus\tilde{S}}|/|W_{S_0\setminus\tilde{S}}|$.

Let (W,S) be a Coxeter system. Let $\mathcal{S}^f(W,S)$ be the family of subsets T of S such that W_T is finite. We note that the empty set is a member of $\mathcal{S}^f(W,S)$. We define a simplicial complex $L(W,S)$ by the following conditions:
(1) the vertex set of $L(W,S)$ is S, and
(2) for each nonempty subset T of S, T spans a simplex of $L(W,S)$ if and only if $T \in \mathcal{S}^f(W,S)$.

For each nonempty subset T of S, $L(W_T,T)$ is a subcomplex of $L(W,S)$. In this paper, $\mathcal{S}^f(W,S)$, $L(W,S)$ and $L(W_T,T)$ are abbreviated to $\mathcal{S}^f$, L and L_T, respectively.

Let (W,S) be a Coxeter system and let $W\mathcal{S}^f$ be the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. The sets $\mathcal{S}^f$ and $W\mathcal{S}^f$ are partially ordered by inclusion. Contractible simplicial complexes $K(W,S)$ and $\Sigma(W,S)$ are defined as the geometric realizations of the partially ordered sets $\mathcal{S}^f$ and $W\mathcal{S}^f$, respectively [5, Section 3; 3]. Here $K(W,S)$ is the cone on the barycentric subdivision of $L(W,S)$. For each subset $T \subset S$, $\Sigma(W_T,T)$ is a subcomplex of $\Sigma(W,S)$. In this paper, $K(W,S)$, $\Sigma(W,S)$ and $\Sigma(W_T,T)$ are abbreviated to K, Σ and Σ_T, respectively. The natural embedding $\mathcal{S}^f \to W\mathcal{S}^f$ defined by $T \mapsto W_T$ induces an embedding $K \to \Sigma$ which we regard as an inclusion. The group W acts on Σ via simplicial automorphism. Then $\Sigma = WK$ and $\Sigma/W \cong K$ ([3,5]). For each $w \in W$, wK is called a *chamber* of Σ. If W is infinite, then Σ is noncompact. It is known that Σ can be cellulated so that the link of each vertex is L [4, Sections 9, 10; 8]. In [8], G. Moussong proved that a natural metric on Σ satisfies the CAT(0) condition. Hence, if W is infinite, Σ can be compactified by adding its ideal boundary $\partial\Sigma$ [4, Section 4; 6]. We note that the natural action of W on Σ is properly discontinuous and cocompact [3,4], and this action induces an action of W on $\partial\Sigma$.

We show the following theorem in Section 4.

Theorem. *Let (W,S) be a Coxeter system and $S_0 \subset S$. Then the following statements are equivalent*:
(1) $\partial\Sigma_{S_0}$ *is* W-*invariant*;
(2) $W = W_{\tilde{S}_0} \times W_{S\setminus\tilde{S}_0}$.

This theorem implies the following corollary.

Corollary. *Let (W,S) be an irreducible Coxeter system. If S_0 is a proper subset of S and $\partial\Sigma_{S_0}$ is nonempty, then $\partial\Sigma_{S_0}$ is not W-invariant.*

2. Lemmas on Coxeter groups

In this section, we prove some lemmas for Coxeter groups which are used later.

Definition 2.1. Let (W,S) be a Coxeter system. For each $w \in W$, we define a subset $S(w)$ of S as

$$S(w){:=}\{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S, we define the following subsets of W:

$$A_T{:=}\{w \in W \mid \ell(wt) > \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S \setminus S(w)\},$$

$$C_T{:=}\{w \in W \mid \ell(wt) < \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S(w)\},$$

$$W^T{:=}\{w \in W \mid S(w) = T\} = C_T \cap A_{S\setminus T}.$$

Let (W,S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ ($s_i \in S$) is said to be *reduced*, if $\ell(w) = l$.

The statements in the following lemma are known or obtained by definition.

Lemma 2.2 (Bourbaki [1] and Davis [5])**.** *Let (W,S) be a Coxeter system.*
(i) *$S(w)$ is empty if and only if $w = 1$, i.e., $W^{\emptyset} = \{1\}$.*
(ii) *For each $w \in W$ and $s \in S$, $\ell(ws)$ equals either $\ell(w)+1$* or *$\ell(w)-1$, and $\ell(sw)$ also equals either $\ell(w)+1$ or $\ell(w)-1$.*
(iii) *For each $T \subset S$ and $w \in W_T$, $\ell_T(w) = \ell(w)$, where $\ell_T(w)$ is the length of w in W_T.*
(iv) *Let $T \subset S$, $w \in W_T$ and $s \in S \setminus T$. Then $\ell(ws) = \ell(sw) = \ell(w)+1$.*
(v) *$(W_{S'})^T \subset W^T$ for each subset S' of S containing T.*
(vi) *Let $w = s_1 \cdots s_l$ be a reduced representation and $T \subset S$. If $w \in A_T$, then $s_i \cdots s_l \in A_T$ for each $1 \leqslant i \leqslant l$.*

Lemma 2.3 (Davis [5, Lemma 1.3]). *Let (W,S) be a Coxeter system, $w \in W$ and $T \subset S$. Then there exists a unique element of shortest length in the coset wW_T. Moreover, the following statements are equivalent*:
(i) *w is the element of shortest length in the coset wW_T*;
(ii) *$w \in A_T$*;
(iii) *$\ell(wu) = \ell(w) + \ell(u)$ for each $u \in W_T$*.

Lemma 2.4. *Let (W,S) be a Coxeter system and $S_0 \subset S$. Then $[W : W_{S_0}] = |A_{S_0}|$.*

Proof. By Lemma 2.3, we have

$$\{wW_{S_0} \mid w \in W\} = \{wW_{S_0} \mid w \in A_{S_0}\}.$$

For each $w_1, w_2 \in A_{S_0}$, $w_1 W_{S_0} = w_2 W_{S_0}$ if and only if $w_1 = w_2$ by Lemma 2.3. Thus $[W : W_{S_0}] = |A_{S_0}|$. □

The following lemma is well-known.

Lemma 2.5. *Let G be a group and let H and K be subgroups of G. If $[G : H] < \infty$, then $[K : H \cap K] < \infty$.*

We investigate $L(W,S)$ in the case W has a parabolic subgroup of finite index.

Lemma 2.6. *Let (W,S) be a Coxeter system and $S_0 \subset S$. If W_{S_0} has finite index in W, then $W_{S \setminus S_0}$ is finite (i.e., $L_{S\setminus S_0}$ is a simplex) and $L = L_{S_0} * L_{S \setminus S_0}$.*

Proof. First we show that $W_{S\setminus S_0}$ is finite. Since W_{S_0} has finite index in W, $\{wW_{S_0} \mid w \in W\} \supset \{wW_{S_0} \mid w \in W_{S\setminus S_0}\}$ is finite. We note that $w_1 W_{S_0} = w_2 W_{S_0}$ if and only if $w_1 = w_2$ for each $w_1, w_2 \in W_{S\setminus S_0}$. Hence $W_{S\setminus S_0}$ is finite.

Next we show that $L = L_{S_0} * L_{S\setminus S_0}$. Let σ and τ be simplexes of L_{S_0} and $L_{S\setminus S_0}$, respectively. Since $[W : W_{S_0}] < \infty$, we have that $[W_{\sigma^0 \cup \tau^0} : W_{\sigma^0}] < \infty$ by Lemma 2.5. Since W_{σ^0} is finite, $W_{\sigma^0\cup\tau^0}$ is finite, i.e., $\sigma * \tau$ is a simplex of L. □

We show the following technical lemma needed later.

Lemma 2.7. *Let (W,S) be a Coxeter system, $T \subset S$ and $t_1, \dots, t_n \in S \setminus T$ ($t_i \neq t_j$ if $i \neq j$). Suppose that $t_i t_{i+1} \neq t_{i+1} t_i$ for any $1 \leqslant i \leqslant n-1$ and $t_n t \neq t t_n$ for any $t \in T$. Then*

$$(W_{S\setminus\{t_1,\dots,t_n\}})^T t_n \cdots t_1 \subset W^{\{t_1\}}.$$

Proof. It is sufficient to show that $S(w(t_n \cdots t_1)) = \{t_1\}$ for each $w \in (W_{S\setminus\{t_1,\dots,t_n\}})^T$. Let $w \in (W_{S\setminus\{t_1,\dots,t_n\}})^T$. Then $w \in A_{S\setminus T}$ and $t_n \cdots t_1 \in W_{S\setminus T}$. Hence, by Lemmas 2.3 and 2.2(iv),

$$\ell(w(t_n \cdots t_1)) = \ell(w) + \ell(t_n \cdots t_1) = \ell(w) + n.$$

By the same argument, we have

$$\ell(w(t_n\cdots t_2t_1)t_1)=\ell(w(t_n\cdots t_2))=\ell(w)+n-1.$$

Hence $\{t_1\}\subset S(w(t_n\cdots t_1))$. To prove the reverse inclusion $S(w(t_n\cdots t_1))\subset\{t_1\}$, we show that $S\setminus\{t_1\}\subset S\setminus S(w(t_n\cdots t_1))$ (i.e., $\ell(w(t_n\cdots t_1)s)=\ell(w)+n+1$ for each $s\in S\setminus\{t_1\}$). Let $s\in S\setminus\{t_1\}$. If $s\notin T$, then

$$\ell(w(t_n\cdots t_1)s)=\ell(w)+\ell((t_n\cdots t_1)s)=\ell(w)+n+1,$$

by $(t_n\cdots t_1)s\in W_{S\setminus T}$ and Lemma 2.3. Suppose that $s\in T$. Let $u:=(t_n\cdots t_1)s$. Then $S(u^{-1})=\{t_n\}$, i.e., $u^{-1}\in A_{S\setminus\{t_n\}}$. Since $w^{-1}\in W_{S\setminus\{t_n\}}$,

$$\ell(wu)=\ell(u^{-1}w^{-1})=\ell(u^{-1})+\ell(w^{-1})=\ell(w)+\ell(u)=\ell(w)+n+1$$

by Lemmas 2.3 and 2.2(iv). Thus $\ell(w(t_n\cdots t_1)s)=\ell(w)+n+1$ for each $s\in S\setminus\{t_1\}$. $\square$

Using Lemmas 2.6 and 2.7, we prove the following lemma which plays a key role in the next section.

Lemma 2.8. *Let (W,S) be an irreducible Coxeter system and S_0 a proper subset of S. If W_{S_0} is infinite, then there exists $t\in S\setminus S_0$ such that $A_{S_0}t\cap W_{S_0}$ is infinite.*

Proof. Let $T_0:=S\setminus S_0$. We define $\{S_i\}$ and $\{T_i\}$ by induction as follows:

$$S_{i+1}:=\{s\in S_i\mid st=ts \text{ for each } t\in T_i\},$$

$$T_{i+1}:=\{s\in S_i\mid st\neq ts \text{ for some } t\in T_i\}=S_i\setminus S_{i+1}.$$

Since, S is finite, $T_n\neq\emptyset$ and $T_{n+1}=\emptyset$ for some n. Then $S=T_0\cup T_1\cup\cdots\cup T_n\cup S_n$ and $W=W_{T_0\cup\cdots\cup T_n}\times W_{S_n}$. Since (W,S) is irreducible, S_n is empty, i.e., $S=T_0\cup T_1\cup\cdots\cup T_n$.

Suppose that $A_{S_0}t\cap W_{S_0}$ is finite for each $t\in S\setminus S_0$. Then we show that $L_{S_k}=L_{T_{k+1}}*L_{S_{k+1}}$ and $L_{T_{k+1}}$ is a simplex for each $k=0,1,\ldots,n-1$.

Let $k\in\{0,1,\ldots,n-1\}$. For each $t_k\in T_k$, we define $T_{k+1}(t_k)$ as

$$T_{k+1}(t_k):=\{s\in S_k\mid t_ks\neq st_k\}.$$

Then $T_{k+1}=\bigcup_{t_k\in T_k}T_{k+1}(t_k)$ by the definition of T_{k+1}.

Let $k\in\{0,1,\ldots,n-1\}$, $t_k\in T_k$ and $T'_{k+1}\subset T_{k+1}(t_k)$. By the definition of $\{T_i\}$, there exist $s_i\in T_i$ $(i=0,\ldots,k-1)$ such that $s_{k-1}t_k\neq t_ks_{k-1}$ and $s_is_{i+1}\neq s_{i+1}s_i$ for any $i=0,\ldots,k-2$. By Lemmas 2.7 and 2.2(v),

$$(W_{S_k})^{T'_{k+1}}t_ks_{k-1}\cdots s_1s_0\subset(W_{S_k\cup\{s_0,\ldots,s_{k-1},t_k\}})^{\{s_0\}}\subset W^{\{s_0\}}\subset A_{S\setminus\{s_0\}}\subset A_{S_0}.$$

Hence $(W_{S_k})^{T'_{k+1}}t_ks_{k-1}\cdots s_1\subset A_{S_0}s_0\cap W_{S_0}$. Since $A_{S_0}s_0\cap W_{S_0}$ is finite, $(W_{S_k})^{T'_{k+1}}$ is finite. Thus $\bigcup_{T'_{k+1}\subset T_{k+1}(t_k)}(W_{S_k})^{T'_{k+1}}$ is finite for each $t_k\in T_k$. We note that

$$\bigcup_{T'_{k+1}\subset T_{k+1}(t_k)}(W_{S_k})^{T'_{k+1}}=\{w\in W_{S_k}\mid S(w)\subset T_{k+1}(t_k)\}$$

$$= \{w \in W_{S_k} \mid S_k \setminus T_{k+1}(t_k) \subset S_k \setminus S(w)\}$$

$$= A_{S_k \setminus T_{k+1}(t_k)} \cap W_{S_k}.$$

Thus, $A_{S_k \setminus T_{k+1}(t_k)} \cap W_{S_k}$ is finite for each $t_k \in T_k$. Since $[W_{S_k} : W_{S_k \setminus T_{k+1}(t_k)}] = |A_{S_k \setminus T_{k+1}(t_k)} \cap W_{S_k}|$ by Lemma 2.4 $[W_{S_k} : W_{S_k \setminus T_{k+1}(t_k)}]$ is finite. By Lemma 2.6, $W_{T_{k+1}(t_k)}$ is finite (i.e., $L_{T_{k+1}(t_k)}$ is a simplex) and $L_{S_k} = L_{T_{k+1}(t_k)} * L_{S_k \setminus T_{k+1}(t_k)}$ for each $t_k \in T_k$.

Let $t_{k+1} \in T_{k+1}$. Since $T_{k+1} = \bigcup_{t_k \in T_k} T_{k+1}(t_k)$, there exists $s_k \in T_k$ such that $t_{k+1} \in T_{k+1}(s_k)$. Then

$$L_{S_k} = L_{T_{k+1}(s_k)} * L_{S_k \setminus T_{k+1}(s_k)} = t_{k+1} * L_{T_{k+1}(s_k) \setminus \{t_{k+1}\}} * L_{S_k \setminus T_{k+1}(s_k)}$$

$$= t_{k+1} * L_{S_k \setminus \{t_{k+1}\}}$$

because $L_{T_{k+1}(s_k)}$ is a simplex. Hence $L_{S_k} = t_{k+1} * L_{S_k \setminus \{t_{k+1}\}}$ for each $t_{k+1} \in T_{k+1}$. Thus $L_{T_{k+1}}$ is a simplex (hence $W_{T_{k+1}}$ is finite) and

$$L_{S_k} = L_{T_{k+1}} * L_{S_k \setminus T_{k+1}} = L_{T_{k+1}} * L_{S_{k+1}}.$$

Here k is an arbitrary element of $\{0, 1, \ldots, n-1\}$.

Then,

$$\begin{aligned} L_{S_0} &= L_{T_1} * L_{S_1} \\ &= L_{T_1} * L_{T_2} * L_{S_2} \\ &= \cdots \\ &= (L_{T_1} * \cdots * L_{T_n}) * L_{S_n} \\ &= L_{T_1} * \cdots * L_{T_n}, \end{aligned}$$

since S_n is empty. Hence L_{S_0} is a simplex, i.e., W_{S_0} is finite. This contradicts the assumption of the infiniteness of W_{S_0}. Therefore, $A_{S_0} t \cap W_{S_0}$ is infinite for some $t \in S \setminus S_0$. □

3. Parabolic subgroups of finite index in Coxeter groups

Using Lemma 2.8, we prove the following theorem.

Theorem 3.1. *Let (W, S) be an irreducible Coxeter system. If W is infinite, then W has no parabolic subgroup of finite index other than W itself.*

Proof. Let S_0 be a proper subset of S. If W_{S_0} is finite, then $[W : W_{S_0}] = \infty$ because W is infinite. Suppose that W_{S_0} is infinite. Then, by Lemma 2.8, $A_{S_0} t \cap W_{S_0}$ is infinite for some $t \in S \setminus S_0$. Hence $[W : W_{S_0}] = |A_{S_0}|$ is infinite by Lemma 2.4. Thus we have that $[W : W_{S_0}] = \infty$ for each proper subset S_0 of S. □

Definition 3.2. Let (W,S) be a Coxeter system. Then there exists a unique decomposition $\{S_1,\dots,S_r\}$ of S such that $W=W_{S_1}\times\cdots\times W_{S_r}$ and each Coxeter system (W_{S_i},S_i) is irreducible (cf. [1; 7, p. 30]). We define $\tilde{S}:=\bigcup\{S_i \mid W_{S_i}$ is infinite$\}$. Then $W_{S\setminus\tilde{S}}$ is finite and $W=W_{\tilde{S}}\times W_{S\setminus\tilde{S}}$. We note that W is finite if and only if $\tilde{S}$ is empty.

We obtain the following corollary from Theorem 3.1.

Corollary 3.3. *The parabolic subgroup $W_{\tilde{S}}$ is the minimal parabolic subgroup of finite index in W.*

Proof. It is clear that $[W:W_{\tilde{S}}]<\infty$. We show that $[W_{\tilde{S}}:W_{S'}]=\infty$ for each proper subset S' of $\tilde{S}$. Let $\{S_i\}$ be the decomposition of $\tilde{S}$ such that $W_{\tilde{S}}=W_{S_1}\times\cdots\times W_{S_n}$ and each (W_{S_i},S_i) is irreducible. For each proper subset S' of $\tilde{S}$, there exists a number i_0 such that $S'\cap S_{i_0}$ is a proper subset of S_{i_0}. Since $(W_{S_{i_0}},S_{i_0})$ is irreducible and $W_{S_{i_0}}$ is infinite, $[W_{S_{i_0}}:W_{S'\cap S_{i_0}}]=\infty$ by Theorem 3.1. Hence $[W_{\tilde{S}}:W_{S'}]=\infty$ by Lemma 2.5. □

The following corollary is obtained immediately from Corollary 3.3.

Corollary 3.4. *Let (W,S) be a Coxeter system and $S_0\subset S$. If W_{S_0} is a parabolic subgroup of finite index in W, then*
(1) $\tilde{S}\subset S_0$,
(2) $W_{S_0}=W_{\tilde{S}}\times W_{S_0\setminus\tilde{S}}$ *and*
(3) $[W:W_{S_0}]=|W_{S\setminus\tilde{S}}|/|W_{S_0\setminus\tilde{S}}|$.

4. The boundaries of CAT(0) spaces on which Coxeter groups act

In this section, we investigate the boundary of a certain CAT(0) space Σ on which a Coxeter group act.

We recall some basic properties of CAT(0) spaces. Details of CAT(0) spaces and their boundaries are found in [2,6]. We say that a metric space (X,d) is a *geodesic space* if for each $x,y\in X$, there exists an isometry $\xi:[0,d(x,y)]\to X$ such that $\xi(0)=x$ and $\xi(d(x,y))=y$ (such ξ is called a *geodesic*). Also a metric space (X,d) is said to be *proper* if every closed metric ball is compact.

Let (X,d) be a geodesic space. Two geodesic rays $\xi,\zeta:[0,\infty)\to X$ are said to be *asymptotic* if there exists a constant N such that $d(\xi(t),\zeta(t))\leqslant N$ for each $t\geqslant 0$.

The following proposition is known.

Proposition 4.1 (cf. Bridson and Haefliger [2] and Ghys and de la Harpe [6]). *Let (X,d) be a proper CAT(0) space.*
(1) *For each two points $x,y\in X$, there exists a unique geodesic segment between x and y in X.*

(2) *X is contractible.*
(3) *For each geodesic ray ξ in X and each point $x_0 \in X$, there exists a unique geodesic ray ξ' issuing from x_0 such that ξ and ξ' are asymptotic.*

Let (X,d) be a proper CAT(0) space and $x_0 \in X$. The *boundary of X with respect to x_0*, denoted by $\partial_{x_0}X$, is defined as the set of all geodesic rays issuing from x_0. Then $X \cup \partial_{x_0}X$ has a natural topology, in which X is an open subspace, and a neighborhood basis for each point $\xi \in \partial_{x_0}X$ is given by the sets

$$U(\xi; r, \varepsilon) = \{x \in X \cup \partial X \mid x \notin B(x_0, r),\ d(\xi(r), \xi_x(r)) < \varepsilon\},$$

where $r, \varepsilon > 0$ and $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0}X$). This is called the *cone topology* on $X \cup \partial_{x_0}X$. It is known that $X \cup \partial_{x_0}X$ is a metrizable compactification of X [2,6].

Let x_0 and x_1 be two points of a proper CAT(0) space X. By Proposition 4.1(3), there exists a unique bijection $\Phi : \partial_{x_0}X \to \partial_{x_1}X$ such that ξ and $\Phi(\xi)$ are asymptotic for each $\xi \in \partial_{x_0}X$. It is known that $\Phi : \partial_{x_0}X \to \partial_{x_1}X$ is a homeomorphism [2,6].

Let X be a proper CAT(0) space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in X. The *boundary of X*, denoted by ∂X, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. By Proposition 4.1(3), for each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0}X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0}X$ for each $x_0 \in X$.

Let (X,d) be a proper CAT(0) space and Γ a group which acts on X by isometries. For each element $\gamma \in \Gamma$ and each geodesic ray $\xi : [0,\infty) \to X$, a map $\gamma\xi : [0,\infty) \to X$ defined by $(\gamma\xi)(t) := \gamma(\xi(t))$ is also a geodesic ray. If geodesic rays ξ and ξ' are asymptotic, then $\gamma\xi$ and $\gamma\xi'$ are also asymptotic. Thus γ induces a homeomorphism of ∂X and Γ acts on ∂X.

Let (W,S) be a Coxeter system and let Σ and K be the proper CAT(0) cell complex and its chamber defined in Section 1, respectively. The definition of the metric of Σ is found in [8,4]. Here each n-cell of Σ is a convex subspace of the n-dimensional Euclidean space and the vertex set of each cell of Σ is the form wW_T with $w \in W$ and $T \in \mathcal{S}^f$ [8,4]. We note that the vertex set of Σ is W, and the 1-skeleton $\Sigma^{(1)}$ is the Cayley graph of W with respect to S with unit edges. For each subset $T \subset S$, $\Sigma_T = \Sigma(W_T, T)$ is a subcomplex of $\Sigma = \Sigma(W,S)$.

Let (W,S) be a Coxeter system. A conjugate to an element of S is called a *reflection* of (W,S). For a reflection wsw^{-1} ($w \in W$ and $s \in S$), its fixed point set $F_{wsw^{-1}} := \{x \in \Sigma \mid (wsw^{-1})x = x\}$ is called the *wall* associated to wsw^{-1}. Here we note that

$$\begin{aligned} F_{wsw^{-1}} &= \{x \in \Sigma \mid (wsw^{-1})x = x\} \\ &= w\{w^{-1}x \in \Sigma \mid sw^{-1}x = w^{-1}x\} \\ &= w\{x \in \Sigma \mid sx = x\} \\ &= wF_s. \end{aligned}$$

It is known that $\Sigma \setminus F_{wsw^{-1}}$ has two components which are interchanged by wsw^{-1}, $W \cap F_{wsw^{-1}} = \emptyset$ and for each $u \in W, uK$ is the closure of the component with u of $\Sigma \setminus \bigcup_{w\in W, s\in S} F_{wsw^{-1}}$ [3,4]. Let C be a cell of Σ such that $C \cap F_{wsw^{-1}} \neq \emptyset$. Then $(wsw^{-1})C = C$ and $C \setminus F_{wsw^{-1}}$ has two components which are interchanged by wsw^{-1}. Since C is a convex subspace of the Euclidean space, every geodesic segment in C cannot intersect $F_{wsw^{-1}} \cap C$ in a subinterval of positive length unless it lies entirely within $F_{wsw^{-1}} \cap C$. By the definition of the metric of Σ, every geodesic segment in Σ cannot intersect $F_{wsw^{-1}}$ in a subinterval of positive length unless it lies entirely within $F_{wsw^{-1}}$.

For each representation $w = s_1 \cdots s_l \in W$, we consider the path

$$P_{s_1,\ldots,s_l} := [e, s_1] \cup [s_1, (s_1 s_2)] \cup \cdots \cup [(s_1 \cdots s_{l-2}), (s_1 \cdots s_{l-1})] \cup [(s_1 \cdots s_{l-1}), w]$$

in $\Sigma^{(1)} \subset \Sigma$, where e is the unit element of W.

We prove the following lemma needed later.

Lemma 4.2. *Let (W,S) be a Coxeter system and N the diameter of K in Σ. Then for each $(e \neq) w \in W$, there exists a reduced representation $w = s_1 \cdots s_l$ such that*

$$d_{\mathrm{H}}(\mathrm{Im}\,\xi_w, P_{s_1,\ldots,s_l}) \leqslant N,$$

where d_{H} is the Hausdorff distance and ξ_w is the geodesic from e to w in Σ.

Proof. Let $(e \neq) w \in W$ and ξ_w the geodesic from e to w in Σ. Then ξ_w intersects at least one wall, and there exist sequences $w_0 = e, w_1, \ldots, w_{n-1}, w_n = w \in W$ and $0 = r_0 < r_1 < \cdots < r_n < r_{n+1} = d(e,w)$ such that

$$\xi_w([r_i, r_{i+1}]) = \mathrm{Im}\,\xi_w \cap w_i K$$

for each $i = 0, 1, \ldots, n$. We note that $w_i \neq w_j$ if $i \neq j$, because K is a convex subspace of Σ.

Let $i \in \{1, \ldots, n\}$, let $w_{i-1}^{-1} w_i = s_1^i \cdots s_{l_i}^i$ be a reduced representation, let $T_i := \{s_1^i, \ldots, s_{l_i}^i\}$ and let F_s be the wall associated to s. Then $\Sigma \setminus F_s$ has two components which are interchanged by s. Let X_s^+ and X_s^- be the two components of $\Sigma \setminus F_s$ with $e \in X_s^+$ and $s \in X_s^-$. Then $\Sigma \setminus F_s = X_s^+ \cup X_s^-$,

$$X_s^+ \cap W = \{u \in W \mid \ell(su) > \ell(u)\},$$

$$X_s^- \cap W = \{u \in W \mid \ell(su) < \ell(u)\}$$

(see [3]). We note that F_s, $X_s^+ \cup F_s$ and $X_s^- \cup F_s$ are convex by Proposition 4.1(1). Since $w_{i-1}^{-1} \xi_w(r_i) \in K \cap w_{i-1}^{-1} w_i K \neq \emptyset$, we have that $T_i \in \mathscr{S}^f$ (cf. [3, Lemma 8.1; 4]). Then

$$w_{i-1}^{-1} \xi_w(r_i) \in K \cap w_{i-1}^{-1} w_i K = \bigcap_{s \in T_i} (K \cap F_s) = \bigcap_{u \in W_{T_i}} uK$$

(see [3]).

Suppose that $w_{i-1}^{-1} \in X_t^-$ for some $t \in T_i$. Since $w_{i-1}^{-1}\xi_w(r_i) \in \bigcap_{s\in T_i} K \cap F_s \subset F_t$, we have that

$$w_{i-1}^{-1}\xi_w([0,r_i]) \subset X_t^- \cup F_t,$$

because $X_t^- \cup F_t$ is convex. On the other hand,

$$w_{i-1}^{-1}\xi_w((r_{i-1}+r_i)/2) \in \operatorname{Int} K \subset X_t^+.$$

This contradiction implies that $w_{i-1}^{-1} \in X_s^+$ for each $s \in T_i$. Hence $\ell(sw_{i-1}^{-1}) > \ell(w_{i-1}^{-1})$, i.e., $\ell(w_{i-1}s) > \ell(w_{i-1})$ for each $s \in T_i$. This means that $w_{i-1} \in A_{T_i}$. By Lemma 2.3,

$$\ell(w_i) = \ell(w_{i-1}s_1^i \cdots s_{l_i}^i) = \ell(w_{i-1}) + \ell(s_1^i \cdots s_{l_i}^i).$$

Thus, we obtain the reduced representation

$$\begin{aligned} w = w_n &= w_{n-1}(s_1^n \cdots s_{l_n}^n),\\ &= w_{n-2}(s_1^{n-1} \cdots s_{l_{n-1}}^{n-1})(s_1^n \cdots s_{l_n}^n),\\ &= \cdots\\ &= (s_1^1 \cdots s_{l_1}^1)(s_1^2 \cdots s_{l_2}^2)\cdots(s_1^{n-1} \cdots s_{l_{n-1}}^{n-1})(s_1^n \cdots s_{l_n}^n). \end{aligned}$$

For each $i \in \{1,\ldots,n\}$ and each $j \in \{1,\ldots,l_i\}$,

$$\xi_w(r_i) \in \operatorname{Im}\xi_w \cap \bigcap_{u\in W_{T_i}} w_{i-1}uK \subset \operatorname{Im}\xi_w \cap w_{i-1}(s_1^i \cdots s_j^i)K \neq \emptyset.$$

Therefore,

$$d_{\mathrm{H}}(\operatorname{Im}\xi_w, P_{s_1^1,\ldots,s_{l_1}^1,\ldots,s_1^n,\ldots,s_{l_n}^n}) \leqslant N. \quad \square$$

Using Lemmas 2.8 and 4.2, we prove the following theorem.

Theorem 4.3. *Let (W,S) be a Coxeter system and $S_0 \subset S$. Then the following statements are equivalent*:
(1) $\partial\Sigma_{S_0}$ *is W-invariant*;
(2) $W = W_{\tilde{S}_0} \times W_{S\setminus\tilde{S}_0}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\partial\Sigma_{S_0}$ is W-invariant. If $S = S_0$ or $\tilde{S}_0 = \emptyset$, then it is clear that $W = W_{\tilde{S}_0} \times W_{S\setminus\tilde{S}_0}$. We suppose that S_0 is a proper subset of S and $\tilde{S}_0$ is not empty (i.e., W_{S_0} is infinite).

We show that $W_{\tilde{S}_0\cup(S\setminus S_0)} = W_{\tilde{S}_0} \times W_{S\setminus S_0}$. Suppose that $W_{\tilde{S}_0\cup(S\setminus S_0)} \neq W_{\tilde{S}_0} \times W_{S\setminus S_0}$. Then there exist $s_0 \in \tilde{S}_0$ and $t_0 \in S\setminus S_0$ such that $s_0t_0 \neq t_0s_0$. Let $\{T_1,\ldots,T_n\}$ be a decomposition of $\tilde{S}_0$ such that $W_{\tilde{S}_0} = W_{T_1} \times \cdots \times W_{T_n}$ and each (W_{T_k}, T_k) is irreducible. Since $s_0 \in \tilde{S}_0 = \bigcup_{k=1}^n T_k$, $s_0 \in T_{k_0}$ for some $k_0 \in \{1,\ldots,n\}$. Then $(W_{T_{k_0}\cup\{t_0\}}, T_{k_0}\cup\{t_0\})$ is irreducible and $W_{T_{k_0}}$ is infinite. By Lemma 2.8, $A_{T_{k_0}}t_0 \cap W_{T_{k_0}}$ is infinite. Since $(A_{T_{k_0}}t_0 \cap W_{T_{k_0}})^{-1}$ is a vertex set of Σ, there exists a sequence $\{v_i\} \subset (A_{T_{k_0}}t_0 \cap W_{T_{k_0}})^{-1}$

which converges to a point $\xi(\infty) \in \partial\Sigma_{T_{k_0}} \subset \partial\Sigma_{S_0}$, where ξ is a geodesic ray issuing from e in Σ. For each $j = 1, 2, \ldots$, there exists a number i_j such that $d(e, v_{i_j}) > j$ and $d(\xi_{v_{i_j}}(j), \xi(j)) < 1$, where $\xi_{v_{i_j}}$ is the geodesic from e to v_{i_j} in Σ. By Lemmas 4.2 and 2.2(vi), for each j, there exists $w_j \in (A_{T_{k_0}} t_0 \cap W_{T_{k_0}})^{-1}$ such that $\ell(v_{i_j}) = \ell(w_j) + \ell((w_j)^{-1} v_{i_j})$ and $d(w_j, \xi(j)) < N + 1$, where $N = \operatorname{diam}(K)$. We note that $\{w_j\}$ converges to $\xi(\infty)$ and $\{t_0 w_j\}$ converges to $t_0\xi(\infty)$.

We show that $t_0\xi(\infty) \notin \partial\Sigma_{S_0}$. Since $d(t_0 w_j, t_0\xi(j)) < N + 1$ and $\Sigma_{S_0} \subset W_{S_0} K$,

$$d(t_0\xi(j), \Sigma_{S_0}) \geqslant d(t_0 w_j, W_{S_0}) - (2N + 1).$$

The group W has the word metric $d_\ell(w, w') := \ell(w^{-1} w')$ for each $w, w' \in W$. The inclusion $W \to \Sigma$ is a quasi-isometric embedding, i.e., there exist constants $\lambda \geqslant 1$ and $\varepsilon \geqslant 0$ such that

$$d(w, w') \geqslant \frac{1}{\lambda} d_\ell(w, w') - \varepsilon$$

for each $w, w' \in W$ (cf. [2, p. 140]). Then

$$\begin{aligned} d(t_0\xi(j), \Sigma_{S_0}) &\geqslant d(t_0 w_j, W_{S_0}) - (2N + 1) \\ &\geqslant \frac{1}{\lambda} d_\ell(t_0 w_j, W_{S_0}) - (2N + \varepsilon + 1) \\ &= \frac{1}{\lambda} \min\{\ell((t_0 w_j)^{-1} u) \mid u \in W_{S_0}\} - (2N + \varepsilon + 1). \end{aligned}$$

Here we note that $(t_0 w_j)^{-1} = (w_j)^{-1} t_0 \in A_{T_{k_0}} \cap W_{T_{k_0}} t_0 \subset A_{S_0}$. Indeed, let $w \in A_{T_{k_0}} \cap W_{T_{k_0}} t_0$ and $s \in S_0$. If $s \in T_{k_0}$ then $\ell(ws) > \ell(w)$, since $w \in A_{T_{k_0}}$. If $s \in S_0 \setminus T_{k_0}$ then $\ell(ws) > \ell(w)$ by $w \in W_{T_{k_0} \cup \{t_0\}}$ and Lemma 2.2(iv). Hence $A_{T_{k_0}} \cap W_{T_{k_0}} t_0 \subset A_{S_0}$. Thus,

$$\begin{aligned} &d(t_0\xi(j), \Sigma_{S_0}) \\ &\quad\geqslant \frac{1}{\lambda} \min\{\ell((t_0 w_j)^{-1} u) \mid u \in W_{S_0}\} - (2N + \varepsilon + 1) \\ &\quad= \frac{1}{\lambda} \min\{\ell((t_0 w_j)^{-1}) + \ell(u) \mid u \in W_{S_0}\} \\ &\qquad -(2N + \varepsilon + 1) \quad \text{by Lemma 2.3(iii)} \\ &\quad= \frac{1}{\lambda} \ell((w_j)^{-1} t_0) - (2N + \varepsilon + 1) \\ &\quad= \frac{1}{\lambda} (\ell(w_j) + 1) - (2N + \varepsilon + 1) \quad \text{by Lemma 2.2(iv).} \end{aligned}$$

Here $\frac{1}{\lambda}(\ell(w_j) + 1) - (2N + \varepsilon + 1) \to \infty$ as $j \to \infty$, i.e., there is no constant M such that $d(t_0\xi(j), \Sigma_{S_0}) < M$ for each j. Hence $t_0\xi(\infty) \notin \partial\Sigma_{S_0}$. This contradicts the assumption of the W-invariantness of $\partial\Sigma_{S_0}$. Thus $W_{\tilde{S}_0 \cup (S \setminus S_0)} = W_{\tilde{S}_0} \times W_{S \setminus S_0}$.

By the definition of $\tilde{S}_0$, $W_{S_0} = W_{\tilde{S}_0} \times W_{S_0 \setminus \tilde{S}_0}$. Therefore

$$W = W_{\tilde{S}_0} \times W_{(S\setminus S_0)\cup(S_0\setminus \tilde{S}_0)} = W_{\tilde{S}_0} \times W_{S\setminus \tilde{S}_0}.$$

(2) $\Rightarrow$ (1): Suppose that $W = W_{\tilde{S}_0} \times W_{S\setminus \tilde{S}_0}$. Then we show that $s(\partial\Sigma_{S_0}) = \partial\Sigma_{S_0}$ for each $s \in S$.

It is clear that $s(\partial\Sigma_{S_0}) = \partial\Sigma_{S_0}$ for each $s \in S_0$.

We prove that $s(\partial\Sigma_{S_0}) = \partial\Sigma_{S_0}$ for each $s \in S\setminus S_0$. Let $s \in S\setminus S_0$ and let $\xi : [0,\infty) \to \Sigma_{\tilde{S}_0}$ be a geodesic ray with $\xi(0) = e$. Then we show that $s\xi(\infty) = \xi(\infty)$. Let $r \in [0,\infty)$. There exists $w_r \in W_{\tilde{S}_0}$ such that $d(\xi(r), w_r) \leqslant N$, where $N = \operatorname{diam}(K)$. Then

$$\begin{aligned} d(\xi(r), s\xi(r)) &\leqslant d(\xi(r), w_r) + d(w_r, sw_r) + d(sw_r, s\xi(r)) \\ &\leqslant N + d(w_r, sw_r) + N \\ &= N + d(w_r, w_r s) + N \\ &= N + 1 + N = 2N + 1, \end{aligned}$$

where the equality $sw_r = w_r s$ follows from $w_r \in W_{\tilde{S}_0}$, $s \in S\setminus S_0 \subset S\setminus \tilde{S}_0$ and the assumption: $W = W_{\tilde{S}_0} \times W_{S\setminus \tilde{S}_0}$. Hence ξ and $s\xi$ are asymptotic, i.e., $s\xi(\infty) = \xi(\infty) \in \partial\Sigma_{\tilde{S}_0}$. Thus

$$s(\partial\Sigma_{\tilde{S}_0}) \subset \partial\Sigma_{\tilde{S}_0} \subset s^{-1}(\partial\Sigma_{\tilde{S}_0}) = s(\partial\Sigma_{\tilde{S}_0}),$$

that is, $s(\partial\Sigma_{\tilde{S}_0}) = \partial\Sigma_{\tilde{S}_0}$. Since $W_{S_0\setminus \tilde{S}_0}$ is finite by the definition of $\tilde{S}_0$, $\Sigma_{S_0\setminus \tilde{S}_0}$ is compact. Hence

$$\partial\Sigma_{S_0} = \partial(\Sigma_{\tilde{S}_0} \times \Sigma_{S_0\setminus \tilde{S}_0}) = \partial\Sigma_{\tilde{S}_0}.$$

Thus $s(\partial\Sigma_{S_0}) = \partial\Sigma_{S_0}$ for each $s \in S \setminus S_0$.

Therefore $\partial\Sigma_{S_0}$ is W-invariant. $\square$

The following corollary is obtained from Theorem 4.3.

Corollary 4.4. *Let (W,S) be an irreducible Coxeter system. If S_0 is a proper subset of S and $\partial\Sigma_{S_0}$ is nonempty, then $\partial\Sigma_{S_0}$ is not W-invariant.*

Proof. Let S_0 be a proper subset of S such that $\partial\Sigma_{S_0}$ is nonempty. Then W_{S_0} is infinite, i.e., $\tilde{S}_0$ is nonempty. Since (W,S) is irreducible, $W \neq W_{\tilde{S}_0} \times W_{S\setminus \tilde{S}_0}$. Hence $\partial\Sigma_{S_0}$ is not W-invariant by Theorem 4.3. $\square$

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References

[1] N. Bourbaki, Groupes et Algebrès de Lie, Masson, Paris, 1981 (Chapters IV–VI).
[2] M.R. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Springer, Berlin, 1999.
[3] M.W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. Math. 117 (1983) 293–324.
[4] M.W. Davis, Nonpositive curvature and reflection groups, preprint, 1994.
[5] M.W. Davis, The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. 91 (2) (1998) 297–314.
[6] E. Ghys, P. de la Harpe (Eds.), Sur les Groups Hyperboliques d'apres Mikhael Gromov, Progr. Math, Vol. 83, Birkhäuser, Boston, MA, 1990.
[7] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.
[8] G. Moussong, Hyperbolic Coxeter groups, Ph.D. Thesis, The Ohio State University, 1988.